

THE SHARP LOG-SOBOLEV INEQUALITY ON A COMPACT INTERVAL

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ABSTRACT. We provide a proof of the sharp log-Sobolev inequality on a compact interval.

1. INTRODUCTION

The Gaussian log-Sobolev inequality, due to A. J. Stam [S, 1959, Eqn. 2.3] or Paul Federbush [F, 1969, Eqn. (14)], although often attributed to L. Gross [G, 1975, Cor. 4.2], played a crucial role in Perelman's [P, 2002] proof of the Poincaré Conjecture. We consider log-Sobolev inequalities for finite Lebesgue measure. F. Maggi [M, 2009] observed that the sharp log-Sobolev inequality on the interval follows from an isoperimetric conjecture of Díaz et al. [DHHT, 2010], which remains open, but provided no proof. We found it in Wang [Wa, 1999], who cited Deuschel and Stroock [DS, 1990], who gave a proof of the sharp log-Sobolev inequality on the circle. We then traced this result back to Emery and Yukich [EY, 1987, p. 1], Rothaus [R1, 1980, Thm. 4.3], and Weissler [We, 1980, Thm. 1]. Our Theorem 2.2 shows that the interval case follows quickly from the circle case.

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2. LOG-SOBOLEV INEQUALITY ON A COMPACT INTERVAL

In considering the isoperimetric problem in sectors of the plane with density r^p , Díaz et al. [DHHT, Cor. 4.24, Conj. 4.18] conjectured the following inequality:

$$(1) \quad \left[\int_0^1 r^q d\alpha \right]^{1/q} \leq \int_0^1 \sqrt{r^2 + (q-1)\frac{r'^2}{\pi^2}} d\alpha,$$

where $1 < q \leq 2$. F. Maggi [M] observed that (1) implies the log-Sobolev inequality of Theorem 2.2 below. Here we observe that Theorem 2.2 follows from the following proposition of Weissler.

Proposition 2.1. [We, Thm. 1] *Let f be a non-negative C^1 function on the circle S^1 of length 1. Suppose $\int_{S^1} f^2 = 1$. Then we have the following sharp inequality:*

$$4\pi^2 \int_{S^1} f^2 \log f \leq \int_{S^1} f'^2.$$

Various proofs are discussed in Section 3.

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Theorem 2.2. *Let f be a non-negative C^1 function on the interval $[0, 1]$. Suppose $\int_0^1 f^2 = 1$. Then we have the following inequality:*

$$(1) \quad \pi^2 \int_0^1 f^2 \log f \leq \int_0^1 f'^2.$$

Proof. Let f be any non-negative C^1 function on $[0, 1]$ such that $\int_0^1 f^2 = 1$. Define a non-negative piecewise C^1 function g on S^1 such that

$$g(x) = \begin{cases} f(2x), & \text{if } 0 \leq x \leq \frac{1}{2} \\ f(2-2x), & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Then $\int_{S^1} g^2 = 1$. By smoothing, Proposition 2.1 applies to g . By simple computation, we have that

$$\int_{S^1} g^2 \log g = \int_0^1 f^2 \log f$$

and

$$\int_{S^1} g'^2 = 4 \int_0^1 f'^2.$$

The conclusion follows. □

Remark 2.3. Feng-Yu Wang [Wa, Ex. 1.2] suggested an alternative proof of 2.2(1), but we don't understand his proof. He considered densities $C_\varepsilon \exp(\varepsilon \cos \pi x)$ and functions $f_\varepsilon = \exp(-\varepsilon \cos \pi x)$, with C_ε chosen to make the integral of f_ε^2 equal to 1. Then f_ε satisfies the differential equation

$$(1) \quad f_\varepsilon'' - \pi \varepsilon \sin \pi x f_\varepsilon' = -\pi^2 f_\varepsilon \log f_\varepsilon.$$

He said that it follows that 2.2(1) holds for those functions and densities with sharp constant π^2 . This might follow if it were known that functions realizing equality exist, but Wang himself [Wa, p. 655] admits that "the author is not sure yet whether there always exists [such a function]." Indeed, in the case of the circle with unit density, there apparently is no such function. Of course, the sharp inequality for density 1 would follow as ε approaches 0.

A similar result holds on any interval for a function with root mean square m :

Corollary 2.4. *Let f be a non-negative C^1 function on the interval $[a, b]$. Suppose*

$$\frac{1}{b-a} \int_a^b f^2 = m^2$$

($m > 0$). Then we have the following inequality:

$$(1) \quad \frac{\pi^2}{(b-a)^2} \left(\int_a^b f^2 \log f - (b-a)m^2 \log m \right) \leq \int_a^b f'^2.$$

Proof. Let f be a non-negative C^1 function on the interval $[a, b]$ such that

$$\frac{1}{b-a} \int_a^b f^2 = m^2 > 0$$

($m > 0$). Define a function g on the interval $[0, 1]$ as

$$g(x) = \frac{1}{m} f((b-a)x + a).$$

Then g is non-negative and C^1 . Moreover, we have

$$\int_0^1 g(x)^2 dx = \int_0^1 \frac{1}{m^2} f((b-a)x+a)^2 dx = \frac{1}{(b-a)m^2} \int_0^a f(y)^2 dy = 1.$$

Therefore, we can apply Theorem 2.2 to the function g . We have

$$(2) \quad \frac{\pi^2}{b-a} \int_0^1 g^2 \log g \leq (b-a) \int_0^1 g'^2.$$

Note that

$$g'(x) = \frac{b-a}{m} f'((b-a)x+a).$$

By direct calculation, we have

$$\int_0^1 g'(x)^2 dx = \frac{(b-a)^2}{m^2} \int_0^1 f'((b-a)x+a)^2 dx = \frac{(b-a)^2}{m^2} \int_a^b f'(x)^2 dx.$$

We also have that

$$\begin{aligned} \int_0^1 g(x)^2 \log g(x) dx &= \frac{1}{m^2} \int_0^1 f((b-a)x+a)^2 \log \frac{f((b-a)x+a)}{m} dx \\ &= \frac{1}{(b-a)m^2} \int_a^b f(x)^2 \log \frac{f(x)}{m} dx \\ &= \frac{1}{(b-a)m^2} \int_a^b f^2 (\log f - \log m) \\ &= \frac{1}{(b-a)m^2} \left(\int_a^b f^2 \log f - (b-a)m^2 \log m \right). \end{aligned}$$

Therefore, by plugging these identities into (2), we have

$$\frac{\pi^2}{(b-a)m^2} \left(\int_a^b f^2 \log f - (b-a)m^2 \log m \right) \leq \frac{b-a}{m^2} \int_a^b f'^2.$$

This is equivalent to the desired inequality (1). □

Corollary 2.4 can be written in the following form:

Corollary 2.5. *Let f be a non-negative C^1 function on the interval $[a, b]$. Suppose*

$$\frac{1}{b-a} \int_a^b f = m > 0.$$

Then we have the following inequality:

$$\frac{2\pi^2}{(b-a)^2} \left(\int_a^b f \log f - m \log m \right) \leq \int_a^b \frac{f'^2}{f}.$$

Proof. Define a non-negative piecewise C^1 function g on the interval $[a, b]$ as $g = \sqrt{f}$. Plugging g into Corollary 2.4 yields the desired result. □

Proposition 2.6. *In Theorem 2.2, π^2 is the best possible constant.*

Proof. For any $0 < \varepsilon < 1$, define

$$f_\varepsilon(x) = \sqrt{1 - \varepsilon^2} + \sqrt{2\varepsilon} \cos \pi x.$$

Then by direct computation, we have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\int_0^1 f_\varepsilon'^2}{\int_0^1 f_\varepsilon^2 \log f_\varepsilon} = \pi^2.$$

Therefore, the constant π^2 cannot be replaced by a larger constant. \square

Remark 2.7. The function $\cos \pi x$ comes from the equality case of a Wirtinger inequality which follows from the log-Sobolev inequality [M].

3. PROOFS OF THE SHARP LOG-SOBOLEV INEQUALITY ON THE CIRCLE

We summarize three proofs of Proposition 2.1 given by Rothaus [R1, Thm. 4.3, 1980], Weissler [We, Thm. 1, 1980], Emery and Yukich [EY, p. 1, 1987], and Deuschel and Stroock [DS, Rmk. 1.14(i)].

3.1. Weissler's Proof. Weissler proved a stronger result than Proposition 2.1 by using Fourier expansion of functions of period 2π .

Proposition 3.1. [We, 1980, Theorem 1] *Let $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be in L^2 and suppose $f(\theta) \geq 0$ almost everywhere. Then*

$$\int f^2 \log f \leq \sum_{n=-\infty}^{\infty} |n| |a_n|^2 + \|f\|_2^2 \log \|f\|_2$$

in the sense that if the right hand side is finite, then so is the left hand side and the inequality holds. ($0^2 \log 0$ is taken to be 0.)

Obviously the above inequality is stronger than the following inequality:

$$\int f^2 \log f \leq \sum_{n=-\infty}^{\infty} |n|^2 |a_n|^2 + \|f\|_2^2 \log \|f\|_2$$

which is equivalent to Proposition 2.1 by change of variables as in Corollary 2.4.

Weissler [We] cited Rothaus's previous 1978 paper [R2] but did not have Rothaus's 1980 paper [R1] where Rothaus gave his proof of Proposition 2.1.

3.2. Rothaus's Proof. Rothaus proved Proposition 2.1 by a variational method. He considered an equivalent problem with a positive parameter ρ [R1, Section 4]. If a related constant b_ρ is zero then the log-Sobolev inequality on the circle with the constant $2/\rho$ holds. Therefore, Proposition 2.1 is equivalent to showing that $b_{1/2\pi^2}$ is zero.

For each b_ρ , he showed that a minimizing function exists, is positive and satisfies a related differential equation [R1, Thm. 4.2]. Moreover, for $\rho > 1/2\pi^2$, the only positive solution to the differential equation is the constant function 1 [R1, Thm. 4.3] and hence b_ρ is zero. Therefore in the limit $b_{1/2\pi^2}$ is zero, and Proposition 2.1 follows.

Rothaus [R1] cited Weissler's paper [We] and said that "A result related to Theorem 6.3 appears in [We]."

3.3. Emery and Yukich's Proof. Emery and Yukich [EY, 1987, p. 1] proved Proposition 2.1 by using estimates deploying the Brownian motion semi-group.

Emery and Yukich [EY] cited both Weissler [We] and Rothaus [R1].

3.4. Deuschel and Stroock's Proof. Deuschel and Stroock considered the log-Sobolev inequality in general spaces with densities. As a special case they proved [DS, Rmk. 1.14(i)] that the log-Sobolev constant for the circle of length 1 with Lebesgue measure is the first eigenvalue of the Laplacian, namely $4\pi^2$ (corresponding to the first eigenfunction $\sin 2\pi x$).

Deuschel and Stroock [DS] cited Emery and Yukich [EY].

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